SYMPLECTIC REDUCTION

Sep 21, 2023

SYMPECTIC QUOTIENTS / HAMILTONIAN REDUCTION

Let G be a compact Lie group acting on a smooth symplectic manifold (M, ω).

WANT: TAKE QUOTIENT OF M BY G IN THE CATEGORY OF SYMPLECTIC MANIFOLDS.

IN PHYSICS, WANT TO REDUCE THE DIMENSION OF THE PHASE SPACE USING SYMMETRIES

FIRST ATTEMPT: TAKE THE SMOOTH QUOTIENT M/G ASSUMING THE ACTION IS FREE AND PROPER.

This doesn't work. One reason is the dimension of the quotient is $\dim M - \dim G$, which can be odd, but symplectic manifolds are even dimensional.

SECOND ATTEMPT:

[MARSDEN, WEINSTEIN, MEYER]

Let (G, M, ω, μ) be a hamiltonian G-space where G is a compact Lie group, (M, ω) is a symplectic manifold, and μ is the corresponding moment map. Assuming that G acts freely on $\mu^{-1}(0)$, we have

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{\iota}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M_{red} := \mu^{-1}(0)/G & & \end{array}$$

SYMPLECTIC QUOTIENT

Then,

(M_{red}, ω_{red}) is a symplectic manifold, where ω_{red} satisfies $\pi^*\omega_{red} = \iota^*\omega$.

(M_{red}, ω_{red}) is called the hamitonian reduction/symplectic quotient of (M, ω) by G.

FIRST CONSIDER $G = S^1 / \mathbb{R}$

Def. Let (M, ω) be a symplectic manifold, and G be a Lie group. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action. The action ψ is a **symplectic action** if

$$\psi : G \longrightarrow \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M),$$

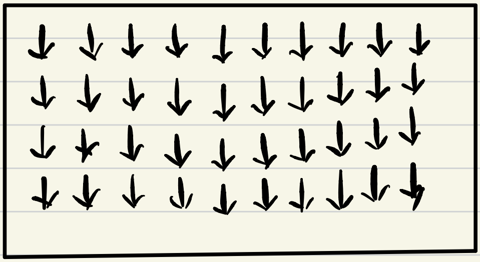
i.e. G acts by symplectomorphisms.

Def. A symplectic action of $S^1 / \mathbb{R}$ on (M, ω) is called hamiltonian if the vector field generated by ψ is hamiltonian. Equivalently, the action of S^1 or $\mathbb{R}$ is **hamiltonian** if there is an $H : M \to \mathbb{R}$ with $dH = \iota_X \omega$, where X is the vector field generated by ψ.

EXAMPLES

(i) $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$

$$X^{\#} = -\frac{\partial}{\partial y_1}$$



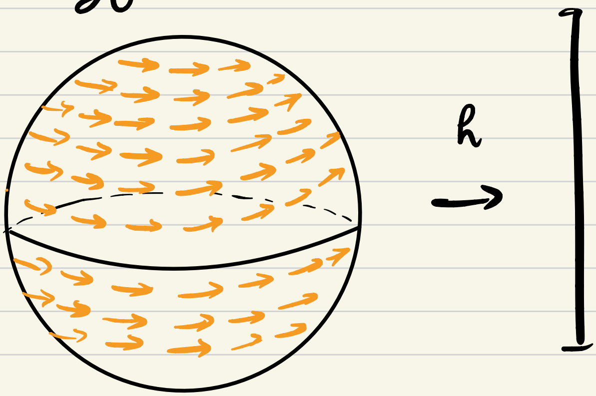
$X^{\#}$ is a hamiltonian with the hamitonian $H = x_1$.

i.e.,

$$\iota_{X^{\#}}\omega = dx_1$$

ii) S^2 with $\omega = d\theta \wedge dh$

$X^\# = \frac{\partial}{\partial\theta}$



$\psi : S^1 \longrightarrow \mathrm{Sympl}(S^2, \omega)$

$t \longmapsto$ rotation by angle t

$X^\#$ is hamitonian with $H = h$.

NOW CASE OF ARBITRARY GROUP

Consider the following diagram:

G acts on (M, ω) by symplectomorphisms

$$\begin{array}{ccc} & & C^\infty(M) \\ & & \downarrow \Phi_\omega \\ \mathfrak{g} & \xrightarrow[\sigma_{inf}]{} & \mathrm{Vect}(M) \end{array}$$

Note that $C^\infty(M)$ is a Lie algebra with bracket $\{f, g\} = \omega(X_f, X_g)$, where X_f and X_g are the vector fields corresponding to f and g.

$\Phi_\omega : C^\infty(M) \to \mathrm{Vect}(M)$ is given by

$$C^\infty(M) \longrightarrow T^*M \xrightarrow[\cong]{} TM$$

$$f \longmapsto df \longrightarrow \omega(X_f, \cdot)$$

$$\sigma_{inf} : \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$$

$$X \longmapsto X^\# \quad \text{where} \quad \frac{d}{dt}(e^{tX} \cdot p)\Big|_{t=0} = X^\#(p)$$

Φ_ω and σ_{inf} are Lie algebra anti-homomorphisms.

$$\begin{tikzcd} & C^\infty(M) \arrow[d, "\Phi_\omega"] \\ \mathfrak{g} \arrow[r, "\sigma_{inf}"'] & \mathrm{Vect}(M) \end{tikzcd}$$

Def. A symplectic action of G on (M, ω) is **hamiltonian** if the map σ_{inf} can be lifted to a Lie algebra homomorphism $\mu^*: \mathfrak{g} \to C^\infty(M)$, called the **comoment map**, such that the following diagram commutes.

$$\begin{tikzcd} & C^\infty(M) \arrow[d, "\Phi_\omega"] \\ \mathfrak{g} \arrow[ur, dashed, "\mu^*"] \arrow[r, "\sigma_{inf}"'] & \mathrm{Vect}(M) \end{tikzcd}$$

MOMENT MAP

Let (M,ω) be a symplectic manifold, G a Lie group, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, and $\psi: G \to \mathrm{Sympl}(M,\omega)$ a symplectic action.

Def. The action ψ is a **hamiltonian action** if there is a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying:

1) $\forall\, X \in \mathfrak{g}$, let

- $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$ (component of μ along X)
- $X^\#$ be the vector field on M generated by $\{e^{tX} \mid t \in \mathbb{R}\}$

Then

$$d\mu^X = \iota_{X^\#}\omega$$

i.e., μ^X is a hamiltonian function for $X^\#$.

2) μ is equivariant wrt. given action ψ on G and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ {\scriptstyle \psi_g}\downarrow & & \downarrow{\scriptstyle Ad_g^*} \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

(M, ω, G, μ) is called a **hamiltonian G-space** and μ is a **moment map**.

MOMENT MAPS CORRESPOND TO COMOMENT MAPS

$$\{\mu : M \to \mathfrak{g}^* \text{ moment maps}\} \longleftrightarrow \{\mu^* : \mathfrak{g} \to C^\infty(M) \text{ comoment maps}\}$$

μ^X:

- $d\mu^X = \iota_{X^\#}\omega$
- $$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\psi_g"'] & \mathfrak{g}^* \arrow[d, "Ad_g^*"] \\ M \arrow[r, "\mu"] & \mathfrak{g}^* \end{tikzcd}$$

$$\longleftrightarrow$$

$$\begin{tikzcd} & C^\infty(M) \arrow[d, "\phi_\omega"] \\ \mathfrak{g} \arrow[ur, dashed, "\mu^*"] \arrow[r, "\sigma_{inf}"'] & V(M) \end{tikzcd}$$

Let μ be a moment map. Take μ^* to be

$$\mu^*(X)\,(p) := \langle \mu(p), X \rangle$$

But $d\mu^*(X) = \iota_{X^\#}\omega$. So, $\phi_\omega \circ \mu^*(X) = X^\# = \sigma_{inf}(X)$

Now for Lie algebra homomorphism: By G equivariance,

$$\langle \mu(g\cdot p), X\rangle = \langle Ad_g^* \cdot \mu(p), X\rangle$$
$$= \langle \mu(p), Ad_{g^{-1}} \cdot X\rangle$$

$\forall\, g \in G, p \in M, X \in \mathfrak{g}$. For, $X, Y \in \mathfrak{g}$, we have

$$0 = \frac{d}{dt}\left(\langle \mu(e^{tY}\cdot p), X\rangle - \langle \mu(p), Ad_{e^{-tY}} \cdot X\rangle\right)$$
$$= \langle d_p\mu(Y^\#), X\rangle - \langle \mu(p), [-Y, X]\rangle$$
$$= \omega(X^\#, Y^\#)(p) - \langle \mu(p), [X, Y]\rangle$$
$$= \{\mu^*(X), \mu^*(Y)\}(p) - \mu^*([X, Y])\,(p)$$

Thus, μ^* is Lie algebra homomorphism.

Conversely, say that μ^* is a Lie algebra homomorphism

<u>WANT:</u> μ is G-equivariant, i.e.

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\psi_g"'] & \mathfrak{g}^* \arrow[d, "Ad_g^*"] \\ M \arrow[r, "\mu"'] & \mathfrak{g}^* \end{tikzcd}$$

Consider the map $\varphi_{p,X} : G \longrightarrow \mathbb{R}$ given by

$$\varphi_{p,X}(g) := \langle \mu(g \cdot p), Ad_g(X) \rangle$$

we want to show that this is constant.
It suffices to prove that the derivative is trivial at the identity e. So,

$$\begin{aligned} d_e \varphi_{p,X}(Y) &= \frac{d}{dt} \langle \mu(e^{tY} \cdot p), Ad_{e^{tY}}(X) \rangle \Big|_{t=0} \\ &= \langle d_p \mu(Y^{\#}), X \rangle + \langle \mu(p), [Y, X] \rangle \\ &= \{\mu^*(X), \mu^*(Y)\}(p) - \mu^*([X,Y])(p) \\ &= 0 \end{aligned}$$

Since μ^* is Lie algebra homomorphism.

BACK TO THE THEOREM

[MARSDEN, WEINSTEIN, MEYER]

Let (G, M, ω, μ) be a hamiltonian G-space where G is a compact Lie group, (M, ω) is a symplectic manifold, and μ is the corresponding moment map. Assuming that G acts freely on $\mu^{-1}(0)$, we have

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{\iota}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M_{red} := \mu^{-1}(0)/G & & \end{array}$$

SYMPLECTIC QUOTIENT

Then,

(M_{red}, ω_{red}) is a symplectic manifold, where ω_{red} satisfies $\pi^* \omega_{red} = \iota^* \omega$.

(M_{red}, ω_{red}) is called the <u>hamitonian reduction/ symplectic quotient</u> of (M, ω) by G.

FIRST INGREDIENT TOWARDS THE PROOF

LEMMA Let $\mathfrak{g}_p$ be the Lie algebra of stabilizer of $p \in M$. Then $d\mu_p : T_pM \to \mathfrak{g}^*$ has

$$\ker d\mu_p = (T_p\mathcal{O}_p)^{\omega_p}$$

$$\operatorname{im} d\mu_p = \mathfrak{g}_p^{\circ}$$

where $\mathcal{O}_p$ is the orbit through p, and $\mathfrak{g}_p^{\circ} = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0, \ \forall\, X \in \mathfrak{g}_p\}$ is the annihilator of $\mathfrak{g}_p$.

PROOF:

$$\begin{aligned}
\ker d\mu_p &= \{ v \in T_pM \mid \langle d\mu_p(v), X\rangle = 0 \ \ \forall\, X \in \mathfrak{g}\} \\
&= \{ v \in T_pM \mid \iota_{X^\#}\omega = 0 \ \ \forall\, X \in \mathfrak{g}\} \\
&= \{ v \in T_pM \mid \omega(X^\#, v) = 0 \ \ \forall\, X \in \mathfrak{g}\} \\
&= (T_p\mathcal{O}_p)^{\omega_p}
\end{aligned}$$

$X \in \mathfrak{g}_p$, $\quad \langle d\mu_p(v), X\rangle = \omega(X^\#, v)$

$$\Rightarrow \langle d\mu_p(v), X\rangle = \omega(0, v) = 0$$

$$\operatorname{Im} d\mu_p \subseteq \mathfrak{g}_p^{\circ}.$$

They have same dimensions.

$$\begin{aligned}
\dim(\operatorname{Im} d\mu_p) &= \dim(T_pM) - \dim(\ker d\mu_p) \\
&= \dim M - \dim(T_p\mathcal{O}_p)^{\omega_p} \\
&= \dim \cancel{M} - (\dim(\cancel{T_pM}) - \dim(T_p\mathcal{O}_p)) \\
&= \dim(T_p\mathcal{O}_p)
\end{aligned}$$

$\underline{\dim \mathfrak{g}_p^0}$

$$0 \to \mathfrak{g}_p^0 \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{g}_p^* \longrightarrow 0$$

$$\dim(\mathfrak{g}_p^0) = \dim(\mathfrak{g}^*) - \dim(\mathfrak{g}_p^*)$$
$$= \dim(G) - \dim(\mathfrak{g}_p)$$
$$= \dim(T_p \mathcal{O}_p)$$

$$\operatorname{Im} d\mu_p = \mathfrak{g}_p^0.$$

LEMMA Let $\mathfrak{g}_p$ be the stabilizer of $p \in M$. Then $d\mu_p : T_pM \to \mathfrak{g}^*$ has

$$\ker d\mu_p = (T_p\mathcal{O}_p)^{\omega_p}$$

$$\operatorname{im} d\mu_p = \mathfrak{g}_p^{\circ}$$

where $\mathcal{O}_p$ is the orbit through p, and $\mathfrak{g}_p^{\circ} = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0,\ \forall X \in \mathfrak{g}_p\}$ is the annihilator of $\mathfrak{g}_p$.

CONSEQUENCES:

- action is locally free at p
  $\Leftrightarrow$ $\mathfrak{g}_p = \{0\}$
  $\Leftrightarrow$ $d\mu_p$ is surjective
  $\Leftrightarrow$ p is a regular point of μ.

- G acts freely on $\mu^{-1}(0)$
  $\Rightarrow$ 0 is a regular value of μ
  $\Rightarrow$ $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to $\dim G$.

- G acts freely on $\mu^{-1}(0)$
  $\Rightarrow$ $T_p\mu^{-1}(0) = \ker d\mu_p$ (for $p \in \mu^{-1}(0)$)
  $\Rightarrow$ $T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements in T_pM.

In particular, the tangent space to the orbit through $p \in \mu^{-1}(0)$ is an isotropic subspace. Hence, orbits in $\mu^{-1}(0)$ are isotropic.

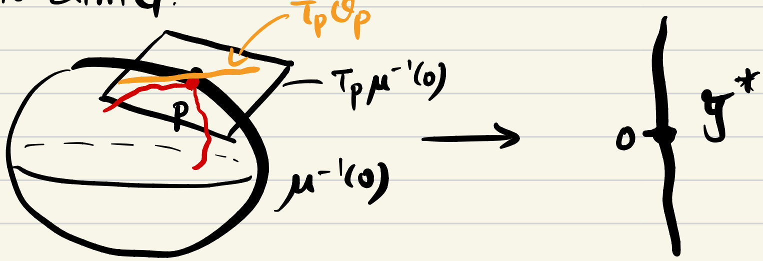
[MARSDEN, WEINSTEIN, MEYER]

Let (G, M, ω, μ) be a hamiltonian G-space where G is a compact Lie group, (M, ω) is a symplectic manifold, and μ is the corresponding moment map. Assuming that G acts freely on $\mu^{-1}(0)$, we have

$$\begin{array}{ccc} \mu^{-1}(0) & \xhookrightarrow{\iota} & M \\ {\scriptstyle \pi}\downarrow & & \\ M_{red} := \mu^{-1}(0)/G & & \end{array}$$

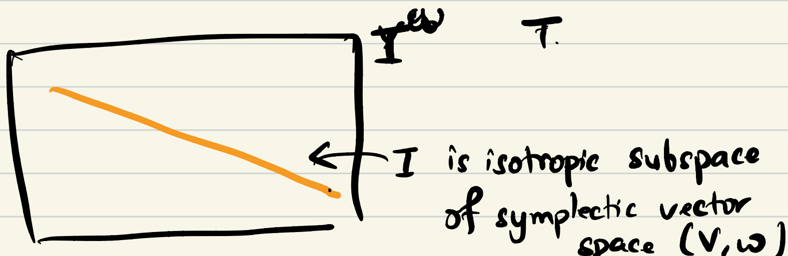
$\leftarrow$ SYMPLECTIC QUOTIENT

Proof. G acts freely on $\mu^{-1}(0)$
$\Rightarrow$ 0 is a regular value of μ
$\Rightarrow$ $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to $\dim G$.



$\operatorname{Ker} d\mu_p = (T_p\mathcal{O}_p)^{\omega_p}$
$\parallel$
$T_p\mu^{-1}(0)$

$\Rightarrow T_p\mathcal{O}_p$ is isotropic



Lemma I^{ω}/I has a canonical symplectic form.

Proof. $[\alpha], [\beta] \in I^{\omega}$. Then
$$\omega([\alpha], [\beta]) = \omega(\alpha, \beta)$$

$\omega([\alpha],[\beta])$

$= \omega(\alpha+i, \beta+j) \qquad i, j \in I \qquad 0$

$= \omega(\alpha,\beta) + \cancel{\omega(\alpha, j)} + \cancel{\omega(i,\beta)} + \cancel{\omega(i,j)}$

$= \omega(\alpha,\beta)$

nondegenerate: Given $\alpha \in I^\omega$

$\omega(\alpha,\beta) = 0 \quad \forall \ \beta \in I^\omega$

$(I^\omega)^\omega = I$

$\Rightarrow \alpha \in I \Rightarrow [\alpha] = 0.$

$T_p\mu^{-1}(0)/T_p\mathcal{O}_p$ has a symplectic form.

Tangent space

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"] & M \\ \mu^{-1}(0)/G = (M_{red}, \omega_{red}) & \end{tikzcd}$$

$$\pi^*\omega_{red} = i^*\omega$$

<u>Closed</u>:

$$\begin{aligned} &\pi^* d\omega_{red} \\ &= d\pi^*\omega_{red} \\ &= d\,i^*\omega \\ &= i^* d\omega \\ &= 0 \end{aligned}$$

π is a submersion.

$\Rightarrow \pi^*$ is injective

$\Rightarrow \quad d\omega_{red} = 0$

$\square$

EXAMPLES

1. Consider the action of S^1 on $(\mathbb{C}^n, \omega)$ given by
$$t \cdot (z_1, \ldots, z_n) = (t z_1, \ldots, t z_n).$$

Let $\mu : \mathbb{C}^n \longrightarrow (\mathfrak{g}^*) \cong \mathbb{R}$ be defined as
$$\mu(z_1, \ldots, z_n) = -\frac{1}{2} \sum_{i=1} |z_i|^2 + \frac{1}{2}$$

we claim that μ is a moment map.
$$\mu^{-1}(0) = \left\{ (z_1, \ldots, z_n) \,\middle|\, \sum |z_i|^2 = 1 \right\}$$
$$= S^{2n-1}$$

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 \cong \mathbb{CP}^{n-1}$$

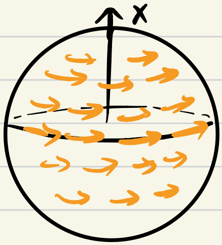
2. We know that $S^1 \curvearrowright S^2$ by rotations is hamiltonian.

Upgrade this to $SO(3) \curvearrowright S^2$ by rotations. This is hamiltonian because of the previous statement. Namely,

$$\mu: S^2 \longrightarrow (\mathfrak{so}(3))^* \cong \mathbb{R}^3 \text{ is inclusion.}$$

Pick $X \in \mathbb{R}^3$. Then

μ^X is rotation around the vector X.



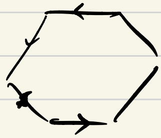
Now take $S^2_{\lambda_1} \times S^2_{\lambda_2} \times \cdots \times S^2_{\lambda_n}$ where

$\lambda_i > 0$ is the radius of the 2-sphere. We can take the moment map to be

$$\mu: S^2_{\lambda_1} \times S^2_{\lambda_2} \times \cdots \times S^2_{\lambda_n} \longrightarrow \mathbb{R}^3$$

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_n$$

Take $\mu^{-1}(0) = \{ (\vec{v}_1, \ldots \vec{v}_n) \mid \vec{v}_1 + \cdots + \vec{v}_n = 0;\ |\vec{v}_i| = \lambda_i \}$



polygons

$\mu^{-1}(0) / SO(3)$ = Moduli space of n-gons upto rotations.